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OBLIQUE DEVIATION AND REFRACTION PRODUCED BY PRISMS.

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The occurrence in the MONTHLY of a few problems[1] in geometrical optics indicates that some of its readers are interested in this subject. As the field is a fairly fertile one for applied mathematics, work in it deserves to be encouraged. Since, in general, there are many discouraging difficulties and pitfalls that the mathematical reader will encounter if he attempts to gain information first hand from the monographs and advanced treatises on geometrical optics, it occurred to the present writer that a relatively short essay dealing with oblique refraction by prisms might be of some interest and perhaps value in the pages of the MONTHLY.

It is but fair to state, at the outset, that none of the results obtained—save the last one—is new. On the other hand, most of the proofs are original and they have been developed with special reference to brevity and rigor.

The only physical law that will be needed is called Snell's law. The complete statement of this law involves two facts which are embodied respectively in the sentence: "The angles of incidence and refraction are coplanar," and in the formula:

$$n_1 \sin a_1 = n_2 \sin a_2. \tag{1}$$

If the ray of light passes from medium 1 into medium 2, then a_1 and a_2 denote respectively the angles of incidence and refraction. n_1 and n_2 symbolize the absolute indices of refraction for the first and second media. The "absolute" index means the ratio of the velocity of light in empty space to the velocity of light in the material dispersive medium in question. The ratio $n_2/n_1 \equiv n$ is called the "relative" index of refraction of medium 2 with respect to medium 1.

In Fig. 1 let the plane IZ suggest the first face of the prism, and let FI and IS indicate respectively the incident ray and the refracted ray. Air and glass may be imagined at the left and right of this plane respectively. Stated broadly, our problem will be to investigate briefly the behavior of the emergent ray beyond the second face of the prism (intentionally omitted from Fig. 1) as the incident ray FI is moved around the point of incidence, I, into all possible positions.

Experience has shown that it is not advantageous to determine the angular positions of the rays by employing a rectangular coördinate frame, direction

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cosines, component vectors, etc. Instead, the analysis and the resulting theorems are very appreciably simplified by introducing a "principal plane," NJ, and by projecting the rays orthogonally on this plane.

Definitions: The line in which the two refracting planes or faces intersect is called the "refracting edge" of the prism. The angle between the refracting planes is named the "refracting angle." Any plane passing through the prism perpendicular to the refracting edge is termed a "principal plane." Hence, a principal plane is parallel to the bases of a right prism of solid geometry. The orthogonal projection, GIG_1, on a principal plane of the oblique ray, FIS, is called the "projected ray." Although both the oblique ray and the projected ray are mathematical fictions,[1] it is conducive to clearness of thought to treat the former as a physical reality (that is, the path along which the energy is transmitted in a non-crystalline medium), and the latter as a purely geometric auxiliary line. The angles which the oblique ray and the projected ray beyond the second or emergence face of the prism make with the corresponding rays before incidence at the first face are called respectively the "deviation of the (oblique) ray" and the "deviation of the projected ray," or the "oblique deviation" and the "projected deviation." The angles, v_1, v_2, $\cdots$,[2] which the segments of the complete oblique ray make with the orthogonal projections on a principal plane will be called the "altitudes" of the rays. Similarly the angles, h_1, h_2, $\cdots$,[2] which the segments of the complete projected ray form with the normals, $NIN_1, \cdots$, to the faces of the prism, at the two points where refraction occurs, will be called the "azimuths" of the oblique ray.

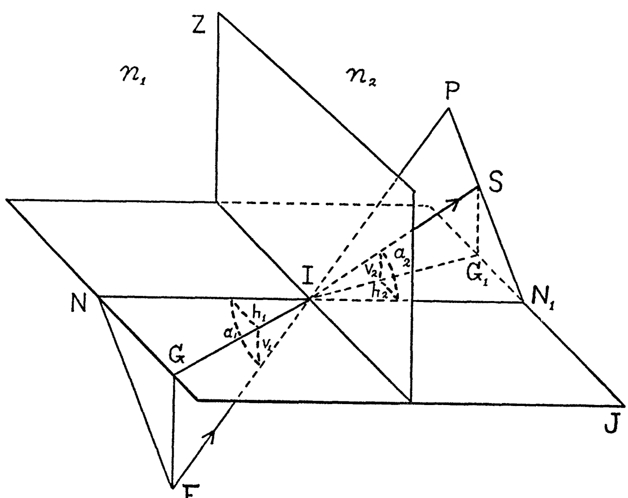

FIG. 1.

In Fig. 1, $NIF \equiv a_1$ and $N_1IS \equiv a_2$ denote respectively the angle of incidence and the angle of refraction of the oblique ray (at the first face of the prism). The outline $FNIN_1P$ indicates the plane of incidence at the first face. The mathematical reader should now be prepared to follow the rest of the paper without undue distraction arising from justifiable unfamiliarity with the highly specialized subject.

For the sake of both consistency and brevity we shall not prove the first formulas of oblique refraction by adding certain highly artificial construction

[1] The wave-length of light is quite finite, diffraction phenomena exist, etc.

[2] h and v to suggest horizontal and vertical respectively.

lines to (the definitional) Fig. 1 and by employing certain theorems of solid geometry (as is done in the older[1] treatises); instead we shall make use of a far more powerful, elegant and general method. This method consists in moving the rays (which are unlocalized vectors), without changing their relative angular directions, until they radiate from the center of a sphere of unit radius. The points where the rays intersect the spherical surface may then be used to define the directions of the rays, and the problems will be reduced primarily to the realm of spherical trigonometry.

Let us now imagine a spherical diagram drawn with the equatorial circle representing the principal plane passing through the point of incidence. Let the points N_1, P, and S indicate respectively the normal to the first face of the prism at the point of incidence, the incident ray, and the refracted ray. Let H mark the pole of the equatorial circle on the same side of the plane of the circle as the points P and S. Imagine great circle arcs connecting H to the points P and S. Let the points of intersection of these two arcs with the equatorial circle be denoted by p and s respectively.

Then the arcs N_1p and N_1s measure the azimuths h_1 and h_2 of the incident and refracted rays. Similarly the arcs pP and sS measure the corresponding altitudes v_1 and v_2.

If a great circle arc be imagined connecting the points N_1 and P it will also pass through the point S because the angle of refraction must lie in the same plane as the angle of incidence (Snell's law in part). Hence the arcs N_1P and N_1S measure respectively the angle of incidence, a_1, and the angle of refraction, a_2.

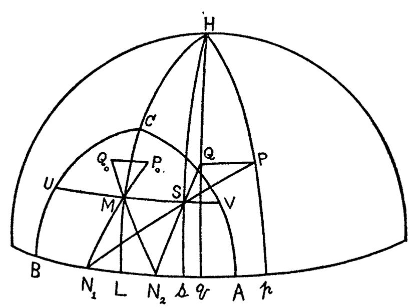

FIG. 2.

We shall now prove that

$$n_1 \sin v_1 = n_2 \sin v_2, \tag{2}$$

$$n_1 \cos v_1 \sin h_1 = n_2 \cos v_2 \sin h_2. \tag{3}$$

In the right spherical triangles N_1pP and N_1sS each member of the following equation is equal to the sine of the common angle pN_1P

$$\frac{\sin v_1}{\sin a_1} = \frac{\sin v_2}{\sin a_2}.$$

Combining this result with equation (1) we obtain formula (2), at a glance. Thus we have the theorem: *The altitudes of the incident and refracted rays obey the trigonometric part of the law of refraction.* Obviously, the angles v_1 and v_2 are not coplanar.

In the right triangles used above, the tangent of the angle pN_1P gives the following equation

$$\frac{\tan v_1}{\sin h_1} = \frac{\tan v_2}{\sin h_2}$$

[1] *Cf.* R. S. Heath, *Geometrical Optics*, p. 21, 1887.

or

$$\sin v_2 \cos v_1 \sin h_1 = \sin v_1 \cos v_2 \sin h_2.$$

Combining this equation with formula (2) we obtain immediately formula (3). If $n_1 \cos v_1$ and $n_2 \cos v_2$ be replaced by n_1' and n_2' respectively, equation (3) assumes the form

$$n_1' \sin h_1 = n_2' \sin h_2,$$

which is algebraically of the same type as formulæ (1) and (2). By analogy, it is customary to speak of $n_1 \cos v_1$ and $n_2 \cos v_2$ as "effective" absolute indices of refraction. This is appropriate since there is no justifiable danger of confusing effective indices with true ones, and n_1' and n_2' play an important rôle in many problems of advanced geometrical optics. This being understood, we may now state the theorem: *The azimuths of the incident and refracted rays obey the law of refraction with effective indices whose ratio is proportional to the ratio of the cosines of the corresponding altitudes.* In this case the planes of the angles h_1 and h_2 coincide.[1]

We are now prepared to follow the path of the oblique ray through and beyond the complete prism.

The points N_2 and Q represent respectively the normal to the second face of the prism at the point of emergence, and the emergent ray. By Snell's law N_2, S, and Q lie on the same great circle. $N_2S \equiv a_3$ is the angle of incidence (within the prism) at the second face. $N_2Q \equiv a_4$ is the angle of refraction of the emergent ray. $qQ \equiv v_4$ is the altitude of the emergent ray, while $N_2q \equiv h_4$ is the azimuth of the same. $N_2s \equiv h_3$ is the azimuth of the incident ray at the second face. $N_1N_2 \equiv \beta$ is the refracting angle of the prism. $QP \equiv D$ is the total deviation of the oblique ray. $qp \equiv D'$ is the total deviation of the projected ray. Angles in the plane N_1sp will be defined as positive when their initial sides have to be rotated in a counterclockwise direction—as viewed from H—in order to bring them into coincidence with the terminal sides. None of the angles measured from normals can exceed $\pi/2$.

Assuming the same medium (usually air) to be in contact with both faces of the prism, formula (2), when applied to the second or emergence face, becomes[2]

$$n_1 \sin v_4 = n_2 \sin v_2. \tag{4}$$

Inspection of equations (2) and (4) shows that $v_4 = v_1$, therefore: *When a prism is surrounded by a single medium, the incident and emergent rays make equal angles with their orthogonal projections on a principal plane.*[3]

It should be noted that this result is independent of the values of the indices

[1] Since only one refracting plane was involved in the preceding proofs it should be clear that the theorems are not restricted to prisms or even to plane interfaces. In Fig. 1, say, IZ may be the tangent plane to any smooth curved surface at the point of incidence. Also the projected ray is a possible path for light of wave-length different from that along the oblique ray.

[2] Obviously $v_3 \equiv v_2$.

[3] This would not hold for the element of a compound prism having air in contact with one face and a different kind of glass with the other face.

of refraction n_1 and n_2. Also it is just what we might have expected, since the elements of the surfaces of the prism which are perpendicular to the principal plane act as a plane-parallel slab of refracting material for the vertical (parallel to the refracting edge) components of the rays, and a slab of this shape (in a single medium) produces no deviation but only a parallel displacement of the rays.

Consequently, in Fig. 2, $qQ = pP = v_1$, and the triangle PHQ is isosceles. Since $\angle\, qHp = D'$, $QP = D$, and $PH = \pi/2 - v_1$ we may imagine a great circle drawn through H so as to bisect QP (and qp). Either of the resulting halves of the triangle PHQ gives

$$\sin \tfrac{1}{2}D = \sin \tfrac{1}{2}D' \cos v_1. \tag{5}$$

Now

$$qp = N_1p - (N_2q + N_1N_2)$$

or

$$D' = h_1 - h_4 - \beta. \tag{6}$$

But, as stated above, $|h_1| \not> \pi/2$, $|h_4| \not> \pi/2$, hence $|h_1 - h_4| \not> \pi$, and relation (6) shows that $D'/2$ must be acute ($\beta > 0$).

For $v_1 \neq 0$, $\cos v_1 < 1$; hence formula (5) shows that $D < D'$ or: *The total deviation of an oblique ray by a triangular prism surrounded by a single medium is less than that of the orthogonal projection of the ray on a principal plane.*

This accounts for the fact that the monochromatic images of a rectilinear slit (spectral lines) formed by any prism spectroscope or spectrograph are appreciably curved. This phenomenon may be observed by looking through any[1] prism toward the vertical boundary of a window, the refracting edge of the prism being vertical also, of course.

Before proceeding farther with the properties of the deviations, attention should be called to a condition which must be fulfilled in order that light may be transmitted directly by a triangular prism. The greatest value that the angle of incidence a_1 can attain is $\pi/2$ ("grazing" incidence) and, when this is the case, equation (1) shows that the greatest value of the angle of refraction a_2 is given by

$$a_2 = \sin^{-1}\frac{1}{n} \equiv c,$$

where $n \equiv n_2/n_1 > 1$. This limiting value of a_2 is called the "critical" angle, c. Similarly, in order that the ray may emerge from the second face of the prism, a_3 must not exceed c. Accordingly, if arcs of small circles be described with N_1 and N_2 as centers and with spherical radii of length c, a lune-like area will be enclosed having the property that the point S,—representative of the internal ray,—must lie within, or on the boundary of, this region in order that transmission without total reflection may occur.[2] In Fig. 2, the upper half of this area is bounded by the arcs AC and BC. If the lower intersection of the limiting small circles be designated as C', then the arcs $C'AC$ and $C'BC$ correspond respectively to grazing incidence and grazing emergence. The points C and C' each signify simultaneous grazing incidence and emergence.

[1] The refracting angle must not be so large as to preclude transmission.

[2] For $n = 1.5$, $c = 41° 48' 37''$, and the greatest value of $\beta = 2c = 83° 37' 14''$.

The most complete and satisfactory method of exploring the domain of transmission $CBC'AC$, for minima of D and D', etc., is to proceed in two steps: (a) to keep the altitude of the ray incident upon the first face of the prism constant while varying the azimuth, and (b) to follow out the behavior of the deviations corresponding to the salient features found under (a) while suitably varying the altitude of the incident ray. More precisely: (a) to keep v_2 [and hence, by (2), v_1 also] constant and explore along a small circle of altitude v_2, such as UV in Fig. 2, and (b) to pick out the interesting points on UV and investigate how the deviations at these points vary as v_2 is changed in such a manner as to sweep over the entire domain from C to BA to C'. The associated analysis will be appreciably simplified by drawing an arc through H and C, cutting the principal plane in L ($N_1L = LN_2 = \beta/2$), and by introducing a new azimuthal angle x, for S, which is reckoned from L as origin. Then equation (3) gives

$$\sin h_1 = n' \sin (x + \tfrac{1}{2}\beta), \tag{7}$$

$$\sin h_4 = n' \sin (x - \tfrac{1}{2}\beta), \tag{8}$$

where

$$n' \equiv n \cos v_2 / \cos v_1 = [n^2 + (n^2 - 1) \tan^2 v_1]^{1/2} > 1 \text{ (for } n > 1).$$

Now let us see what happens along UV. Keeping n' constant while differentiating equations (6), (7), and (8) with respect to x, we find

$$\frac{\partial D'}{\partial x} = \frac{\partial h_1}{\partial x} - \frac{\partial h_4}{\partial x},$$

$$\cos h_1 \frac{\partial h_1}{\partial x} = n' \cos (x + \tfrac{1}{2}\beta),$$

$$\cos h_4 \frac{\partial h_4}{\partial x} = n' \cos (x - \tfrac{1}{2}\beta),$$

whence

$$\frac{\partial D'}{\partial x} = n'[\cos (x + \tfrac{1}{2}\beta)/\cos h_1 - \cos (x - \tfrac{1}{2}\beta)/\cos h_4].$$

Omitting the limiting cases $h_1 = \pi/2$ and $h_4 = \pi/2$ we may multiply and divide by the sum of the two fractions within the bracket, thus obtaining

$$\frac{\partial D'}{\partial x} = \frac{n'[\cos^2 (x + \tfrac{1}{2}\beta)/\cos^2 h_1 - \cos^2 (x - \tfrac{1}{2}\beta)/\cos^2 h_4]}{\cos (x + \tfrac{1}{2}\beta)/\cos h_1 + \cos (x - \tfrac{1}{2}\beta)/\cos h_4}.$$

By elementary trigonometric transformations, the last equation may be readily reduced to

$$\frac{\partial D'}{\partial x} = \frac{2n'(n'^2 - 1) \sin \beta \sin x \cos x}{\cos h_1 \cos h_4(\cos h_1 \cos h_3 + \cos h_2 \cos h_4)}. \tag{9}$$

Since $\beta \not> \pi$, $n' > 1$, and none of the remaining angles can exceed $\pi/2$, we see that the sign of the derivative is the same as that of $\sin x$. For any two

permissible values of x having the same magnitude but opposite signs, the derivative also has equal numerical values with opposite signs. Moreover it vanishes when $x = 0$. Consequently, the values of the deviation of the projected ray, D', are symmetrical with respect to the great circle $x = 0$, and they have a minimum both in the algebraic and in the arithmetic sense when, and only when, $x = 0$. Inspection of equations (6), (7), and (8) shows that no exception to the last statement arises when either h_1 or h_4 equals $\pi/2$. When $x = 0$, $h_2 = - h_3 = \beta/2$, and

$$h_1 = - h_4 = \sin^{-1} (n' \sin \tfrac{1}{2}\beta).$$

Since v_1 is being kept constant, equation (5) shows that D has precisely the same minimum properties as D'.

These results may be interpreted in terms of the actual prism in the following manner. Imagine a plane diaphragm constructed in the interior of the prism and fulfilling three conditions; (a) of passing through the point of incidence at the first face, (b) of being parallel to the refracting edge, and (c) of making equal angles $(\pi - \beta)/2$ with the refracting faces of the prism, that is, crossing the prism symmetrically. Then, as the altitude of the ray incident upon the first face is kept constant while its azimuth is varied, the oblique deviation and the projected deviation will both have minimum values at the instant when the internal ray lies in the plane of symmetry just defined. The winglike planes containing the minimum positions of the incident and emergent rays, which may be imagined outside of the prism, will also be symmetrically situated, but the common value of the equal angles, $\pi/2 - \sin^{-1} [n' \sin (\beta/2)]$, that these planes make with the refracting faces of the prism will not be independent of the altitude of the incident ray, hence their positions will be altered if a new constant value for v_1 be taken, since n' is a function of v_1.

With this interpretation of the term "symmetrical," we may now state the theorem: *When the altitude of the internal ray, or of the external rays, is kept constant, both the oblique deviation and the projected deviation have least values when, and only when, the rays are situated symmetrically with respect to the refracting faces of the prism.*

For the sake of completeness it may be remarked that, when the altitude of the incident ray is kept constant while the azimuth is varied, the incident ray and the internal ray each describe portions of the lateral surfaces of right circular cones having as common axis a line (in the incidence face) passing through the point of incidence and parallel to the refracting edge, the semi-apical angles of the cones being respectively the complements of the altitudes v_1 and v_2.

Let D_0 and D_0' denote respectively the minimum values of D and D' which occur when x equals zero and v_1 is constant. The next step will be to keep x zero and to investigate the behavior of the (partial) minima D_0 and D_0' as the point S moves along the great circle arc from C to L to C'. The independent variable is now v_2. With direct reference to the prism, our immediate problem is to cause the altitudes of the three parts of the complete ray to vary in such a

manner as to maintain the above mentioned symmetry and to investigate the way in which the minima of deviation depend upon the values of the altitudes. Under these circumstances the internal ray will be constrained to move up or down (fanwise) in the transverse diaphragm introduced earlier. It will be convenient to take up D_0 before D_0'.

Consider the halves of the isosceles triangles N_1MN_2 and P_0MQ_0, and let each of the four equal acute angles having the common vertex M be denoted by e. Since $Q_0P_0/2 = D_0/2$, $N_1L = \beta/2$, $MP_0 = a_1 - a_2$, and $N_1M = a_2$, we have

$$\sin e = \sin \tfrac{1}{2}D_0/\sin (a_1 - a_2) = \sin \tfrac{1}{2}\beta/\sin a_2$$

or

$$\sin \tfrac{1}{2}D_0 = (\sin a_1 \cos a_2 - \cos a_1 \sin a_2) \sin \tfrac{1}{2}\beta/\sin a_2.$$

As equation (1) is

$$\sin a_1 = n \sin a_2$$

it is clear that

$$\cos a_1 = + [n^2 \cos^2 a_2 - (n^2 - 1)]^{1/2}$$

and

$$\sin \tfrac{1}{2}D_0 = \{n \cos a_2 - [n^2 \cos^2 a_2 - (n^2 - 1)]^{1/2}\} \sin \tfrac{1}{2}\beta. \tag{10}$$

As $LM = v_2$, the right triangle N_1LM gives

$$\cos a_2 = \cos \tfrac{1}{2}\beta \cos v_2, \tag{11}$$

which, combined with the process of rationalizing the braced expression in equation (10), leads to

$$\sin \tfrac{1}{2}D_0 = \frac{(n^2 - 1) \sin \tfrac{1}{2}\beta}{n \cos \tfrac{1}{2}\beta \cos v_2 + [(n \cos \tfrac{1}{2}\beta \cos v_2)^2 - (n^2 - 1)]^{1/2}}. \tag{12}$$

Since $\cos v_2$ increases as v_2 decreases numerically, we see from equation (12) that D_0 attains its least value when $v_2 = 0$, that is, when the entire ray lies in the principal plane. This does not show explicitly that D_0 has a minimum in the algebraic sense when $v_2 = 0$, for, a cusp of the first species might be present at the lowest point of a surface constructed on a rectangular coördinate frame and exhibiting D as a function of x and v_2.

In order to show that D_0 has a true algebraic minimum when $v_2 = 0$ we must form the derivative of D_0 with respect to v_2. It will be found, without difficulty, by differentiating equations (10) and (11), and then eliminating da_2/dv_2, that

$$\frac{dD_0}{dv_2} = \frac{n \sin \beta \sin v_2}{\cos \tfrac{1}{2}D_0} \left\{ \frac{n \cos \tfrac{1}{2}\beta \cos v_2}{[(n \cos \tfrac{1}{2}\beta \cos v_2)^2 - (n^2 - 1)]^{1/2}} - 1 \right\}.$$

The expression enclosed by braces in the last equation, $\sin \beta$, and $\cos \tfrac{1}{2}D_0$ are always positive, hence the derivative has the same sign as $\sin v_2$. The derivative also approaches zero simultaneously with v_2. It should be evident, therefore, that D_0 has an algebraic minimum when, and only when, $v_2 = 0$.

When $x = 0$, equation (5) may be written

$$\sin \tfrac{1}{2}D_0' = \sin \tfrac{1}{2}D_0 \sec v_1,$$

which shows, at a glance, that D_0' has an algebraic minimum simultaneously with D_0, for both $\sin(D_0/2)$ and $\sec v_1$ decrease as v_2 becomes smaller.

The whole region bounded by $CBC'AC$ has now been explored and we have proved the theorem: *The deviation produced by direct transmission through a triangular prism has the smallest value, which is also an absolute algebraic minimum, when the ray lies in a principal plane and makes equal angles with the refracting faces.* This is subject to the qualifications that the prism is surrounded by one medium and that its relative index of refraction is greater than unity.

As far as the writer is aware, the following generalization of formula (5) is entirely new. Suppose we have k prisms with their refracting edges parallel, and surrounded by one and the same medium. The prisms must be so situated, of course, as to permit the passage of light through the system.

The hypothesis of a single surrounding medium causes v_1 to be constant throughout the entire prism train, quite independently of the various indices of refraction of the materials composing the different prisms. [See equation (4) and the associated context.] Also, the refracting angles need not bear any particular relation to one another. Since the emergent ray from any prism (save the last) becomes the incident ray of the next succeeding prism, the representative points P and Q of Fig. 2 will be distributed along a small circle arc of altitude v_1. Accordingly, the separate deviations produced by the individual prisms will form collectively a fluting of great circle arcs having salient points on the small circle just mentioned. With these flutings, however, we are not directly concerned, for the single great circle arc connecting the terminus of the last or kth emergent ray to that of the first incident ray will represent the total deviation D produced by the entire train of prisms. Incidentally, $D < \sum_1^k D_j$.

As in the case of a single prism, so also here, the triangle PHQ,—where P and Q now mark the extreme ends of the fluting,—will be isosceles, so that

$$\sin \tfrac{1}{2}D = \sin \tfrac{1}{2}D' \cos v_1. \qquad (5')$$

Since the segments of the oblique ray are all projected orthogonally on the common principal plane, the following sub-relation between the total projected deviation and the separate projected deviations obviously holds:

$$D' = \sum_1^k D_j'.$$

Bibliographical and Historical Notes: The most reliable and elegant presentation of geometrical optics in English is the second edition of J. P. C. Southall's *Principles and Methods of Geometrical Optics*, New York (1913). The most complete general discussion of prisms was written by H. Konen and published in H. Kayser's *Handbuch der Spectroscopie*, vol. I, pp. 253–394, (1900). On page 258 may be found a very complete list of references to proofs of the theorem of minimum deviation *in a principal plane*.

This theorem dates from the work of Newton. The reference given by Konen

is "Lect. opt. London 1729. P. I., Sect. II, Art. 31." For oblique deviation, the fact that $v_4 = v_1$ was published by A. Bravais in 1845, *Journal de l'Ecole Polytechnique*, vol. 18, 30^e Cahier, p. 79. The priority for formula (5) seems to belong to Mascart. This equation is given correctly in his *Traité d'Optique*, vol. 1, p. 84 (1889). By drawing an incorrect diagram, R. S. Heath (*l.c.* p. 32) derived the formula

$$\cos \tfrac{1}{2}D = \cos \tfrac{1}{2}D' \cos v_1.$$

This was copied by almost all later writers regardless of the fact that J. Larmor called attention to the error in the *Proceedings of the Cambridge Philosophical Society*, vol. 9, p. 108 (1896). An unbiased discussion of this matter (including the part which the present writer has taken) is given in Southall's treatise, p. 127. Finally, Konen in attempting to generalize formula (5) extended the error even farther by stating that the cosine equation still held (*l.c.* p. 267).

THE FIRST WORK ON MATHEMATICS PRINTED IN THE NEW WORLD.[1]

By DAVID EUGENE SMITH, Columbia University.

I. General Description. If the student of the history of education were asked to name the earliest work on mathematics published by an American press, he might, after a little investigation, mention the anonymous arithmetic that was printed in Boston in the year 1729. It is now known that this was the work of Isaac Greenwood who held for some years the chair of mathematics in what was then Harvard College. If he should search the records still further back, he might come upon the American reprint of Hodder's well-known English arithmetic, the first textbook on the subject, so far as known, to appear in our language on this side of the Atlantic. If he should look to the early Puritans in New England for books of a mathematical nature, or to the Dutch settlers in New Amsterdam, he would look in vain; for, so far as known, all the colonists in what is now the United States were content to depend upon European textbooks to supply the needs of the relatively few schools that they maintained in the seventeenth century.

The earliest mathematical work to appear in the New World, however, antedated Hodder and Greenwood by more than a century and a half. It was published long before the Puritans had any idea of migrating to another continent, and fifty years before Henry Hudson discovered the river that bears his name. Of this work, known as the *Sumario Compendioso*, there remain perhaps only four copies, and it is desirable, not alone because of its rarity but because of its im-

[1] Address delivered before The Mathematical Association of America, and the section on the History of Science of the American Association for the Advancement of Science, at the University of Chicago, December 28, 1920. The extracts are from a fac-simile reprint of the original work soon to published by Ginn & Company, Boston, with translation and notes by Professor Smith.